

# Ensemble Averages when $\beta$ is a Square Integer

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## Abstract

We give a hyperpfaffian formulation of partition functions and ensemble averages for Hermitian and circular ensembles when  $L$  is an arbitrary integer and  $\beta = L^2$  and when  $L$  is an odd integer and  $\beta = L^2 + 1$ .

**Keywords:** Random matrix theory, partition function, Pfaffian, hyperpfaffian, Selberg integral

For many ensembles of random matrices, the joint probability density of eigenvalues has the form

$$\Omega_N(\boldsymbol{\lambda}) = \frac{1}{Z_N N!} \left\{ \prod_{n=1}^N w(\lambda_n) \right\} \prod_{m < n} |\lambda_n - \lambda_m|^\beta, \quad \boldsymbol{\lambda} \in \mathbb{R}^N \quad (0.1)$$

where,  $\beta > 0$ ,  $w : \mathbb{R} \rightarrow [0, \infty)$  is a weight function and  $Z_N$  is a normalization constant called the *partition function*. The parameter  $\beta$  is often taken to be 1, 2 or 4, since for these values the correlation functions (*viz.* marginal probabilities) have a determinantal or Pfaffian form.

The partition function is given explicitly by

$$Z_N = Z_N^\nu = \frac{1}{N!} \int_{\mathbb{R}^N} \prod_{m < n} |\lambda_n - \lambda_m|^\beta d\nu^N(\boldsymbol{\lambda}), \quad (0.2)$$

where  $d\nu(\lambda) = w(\lambda)d\lambda$  and  $\nu^N$  is the resulting product measure on  $\mathbb{R}^N$ . However, for now, we will take  $\nu$  to be arbitrary, with the restriction that  $0 < Z_N < \infty$ , and we will suppress the notational dependence of  $Z_N$  on  $\nu$  except when necessary to allay confusion. Ensemble averages of functions which are multiplicative in the eigenvalues can be expressed as integrals of the form (0.2) where  $\nu$  is determined by the function on the eigenvalues, the weight function of the ensemble and the underlying reference measure.

In the case where  $\beta = 2\gamma$  and

$$d\nu(\lambda) = \lambda^{a-1}(1-\lambda)^{b-1} \mathbf{1}_{[0,1]}(\lambda) d\lambda; \quad a, b, \gamma \in \mathbb{C},$$

$S_N(\gamma, a, b) = N!Z_N$  is the (now) famous Selburg integral [19, 20], and in this case

$$S_N(\gamma, a, b) = \prod_{n=0}^{N-1} \frac{\Gamma(a+n\gamma) \Gamma(b+n\gamma) \Gamma(1+(n+1)\gamma)}{\Gamma(a+b+(N+n-1)\gamma) \Gamma(1+\gamma)}.$$

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The case where

$$d\nu(\lambda) = \frac{1}{\sqrt{2\pi}} e^{-\lambda^2/2} \quad \text{and} \quad \beta = 2\gamma,$$

$F_N(\gamma) = N!Z_N$  is called the Mehta integral [15, 17], and has the evaluation

$$F_N(\gamma) = \prod_{n=1}^N \frac{\Gamma(1+n\gamma)}{\Gamma(1+\gamma)}.$$

This evaluation follows the same basic idea as that of the Selberg integral. For an interesting history of these evaluations, see [9].

The purpose of this note is not to provide further explicit evaluations of for particular choices of  $\nu$ , but rather to show that when  $\beta$  is a square integer (or adjacent to a square integer) that  $Z_N$  can be expressed as a hyperpfaffian. Expressing ensemble averages as a determinant or Pfaffian is the first step to demonstrating the solvability of an ensemble; that is writing the correlation functions, gap probabilities and other quantities of interest in terms of determinants or Pfaffians formed from a kernel associated to the particulars of the ensemble [22, 21, 5]. Pfaffian formulations for  $Z_N$  are known when  $\beta = 1$  and  $\beta = 4$ , and the results here generalize those formulas. To be specific, we show that when  $L$  is odd and  $\beta = L^2$  or  $L^2 + 1$ , then  $Z_N$  can be written as the hyperpfaffian of a  $2L$ -form, the coefficients of which are formed from double integrals of products of  $L \times L$  Wronskians of monic polynomials in a manner which generalizes the well known  $\beta = 1$  and  $\beta = 4$  cases. When  $L$  is even,  $Z_N$  can be written as a hyperpfaffian of an  $L$ -form, the coefficients of which are integrals (with respect to the measure  $\nu$ ) of Wronskians of any complete family of monic polynomials. This generalizes the situation when  $\beta = 4$ . Similar results for  $L$  even were given in [14].

We will also give a similar characterization of ensemble averages over circular ensembles.

## 1 Pfaffians, Hyperpfaffians and Wronskians

### 1.1 Pfaffians and Hyperpfaffians

Suppose  $V = \mathbb{R}^{NL}$  with basis  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{NL}$ . For each increasing function  $\mathbf{t}$ , from  $\underline{L} = \{1, 2, \dots, L\}$  into  $\underline{NL} = \{1, 2, \dots, NL\}$ , we write

$$\epsilon_{\mathbf{t}} = \mathbf{e}_{\mathbf{t}(1)} \wedge \mathbf{e}_{\mathbf{t}(2)} \wedge \dots \wedge \mathbf{e}_{\mathbf{t}(L)},$$

so that  $\{\epsilon_{\mathbf{t}} \mid \mathbf{t} : \underline{L} \nearrow \underline{NL}\}$  is a basis for  $\Lambda^L V$ . The volume form of  $\mathbb{R}^{NL}$  is given by

$$\epsilon_{\text{Vol}} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \dots \wedge \mathbf{e}_{NL}.$$

The  $N$ -fold wedge product of an  $L$ -form  $\omega$  is a constant times  $\epsilon_{\text{Vol}}$ . This constant is (up to a factor of  $N!$ ) the *hyperpfaffian* of  $\omega$ . Specifically, for  $\omega \in \Lambda^L V$ , we define  $\text{PF } \omega$  by

$$\frac{\omega^{\wedge N}}{N!} = \frac{1}{N!} \underbrace{\omega \wedge \omega \wedge \dots \wedge \omega}_N = \text{PF } \omega \cdot \epsilon_{\text{Vol}}.$$

When  $L = 2$ , the hyperpfaffian generalizes the notion of the Pfaffian of an antisymmetric matrix: If  $\mathbf{A} = [a_{m,n}]$  is an antisymmetric  $2N \times 2N$  matrix, then the Pfaffian of  $\mathbf{A}$  is given

by

$$\text{Pf } \mathbf{A} = \frac{1}{2^N N!} \sum_{\sigma \in S_{2N}} \text{sgn } \sigma \prod_{n=1}^N a_{\sigma(2n-1), \sigma(2n)}.$$

We may identify  $\mathbf{A}$  with the 2-form

$$\alpha = \sum_{m < n} a_{m,n} \mathbf{e}_m \wedge \mathbf{e}_n.$$

It is an easy exercise to show that  $\text{Pf } \mathbf{A} = \text{PF } \alpha$ . There exists a formula for the hyperpfaffian as a sum over the symmetric group, and in fact, the original definition of the hyperpfaffian was given as such a sum [14, 3].

## 1.2 Wronskians

A *complete* family of monic polynomials is an  $NL$ -tuple of polynomials  $\mathbf{p} = (p_n)_{n=1}^{NL}$  such that each  $p_n$  is monic and  $\deg p_n = n - 1$ . Given an increasing function  $\mathbf{t} : \underline{L} \nearrow \underline{NL}$ , we define the  $L$ -tuple  $\mathbf{p}_{\mathbf{t}} = (p_{\mathbf{t}(\ell)})_{\ell=1}^L$ . And, given  $0 \leq \ell < L$  we define the modified  $\ell$ th differentiation operator by

$$D^\ell f(x) = \frac{1}{\ell!} \frac{d^\ell f}{dx^\ell} \quad (1.1)$$

The *Wronskian* of  $\mathbf{p}_{\mathbf{t}}$  is then defined to be

$$\text{Wr}(\mathbf{p}_{\mathbf{t}}; x) = \det [D^{\ell-1} p_{\mathbf{t}(\ell)}(x)]_{n,\ell=1}^L.$$

The Wronskian is often defined without the  $\ell!$  in the denominator of (1.1); this combinatorial factor will prove convenient in the sequel. The reader has likely seen Wronskians in elementary differential equations, where they are used to test for linear dependence of solutions.

## 2 Statement of Results

For each  $x \in \mathbb{R}$ , we define the  $L$ -form  $\omega(x) \in \Lambda^L V$  by

$$\omega(x) = \sum_{\mathbf{t} : \underline{L} \nearrow \underline{NL}} \text{Wr}(\mathbf{p}_{\mathbf{t}}; x) \epsilon_{\mathbf{t}}. \quad (2.1)$$

This form clearly depends on the choice of  $\mathbf{p}$ , though we will suppress this dependence. We may arrive at an  $L$ -form with constant coefficients by integrating the coefficients of the form with respect to  $d\nu$ . In this situation we will write

$$\int_{\mathbb{R}} \omega(x) d\nu(x) = \sum_{\mathbf{t} : \underline{L} \nearrow \underline{NL}} \left\{ \int_{\mathbb{R}} \text{Wr}(\mathbf{p}_{\mathbf{t}}; x) d\nu(x) \right\} \epsilon_{\mathbf{t}}.$$

Notice that we are not integrating the form in the sense of integration on manifolds, but rather formally extending the linearity of the integral over the coefficient functions of  $\omega(x)$  to arrive at an  $L$ -form with constant coefficients.

To deal with the  $N$  odd case, we set  $V' = \mathbb{R}^{(N+1)L}$  and define the basic  $L$ -form  $\epsilon' = \mathbf{e}_{NL+1} \wedge \mathbf{e}_{NL+2} \wedge \cdots \wedge \mathbf{e}_{NL+L}$ .

**Theorem 2.1.** Suppose  $L$  and  $N$  are positive integers, and let  $\omega(x)$  be the  $L$ -form given as in (2.1) for any complete family of monic polynomials  $\mathbf{p}$ . Then,

1. if  $\beta = L^2$  is even,

$$Z_N = \text{PF} \left( \int_{\mathbb{R}} \omega(x) d\nu(x) \right);$$

2. if  $\beta = L^2$  is odd and  $N$  is even,

$$Z_N = \text{PF} \left( \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \omega(x) \wedge \omega(y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right);$$

3. if  $\beta = L^2$  is odd and  $N$  is odd,

$$Z_N = \text{PF} \left( \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \omega(x) \wedge \omega(y) \text{sgn}(y-x) d\nu(x) d\nu(y) + \int_{\mathbb{R}} \omega(x) \wedge \epsilon' d\nu(x) \right);$$

4. if  $\beta = L^2 + 1$  is even and  $N = 2M$  is even,

$$Z_N = \text{PF} \left( \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \omega(x) \wedge \omega(y) \left[ \frac{(y^M - x^M)^2}{y-x} \right] d\nu(x) d\nu(y) \right).$$

More explicitly, when  $\beta = L^2$  is even we have

$$\int_{\mathbb{R}} \omega(x) d\nu(x) = \sum_{\mathbf{t}: \underline{L} \nearrow \underline{NL}} \int_{\mathbb{R}} \text{Wr}(\mathbf{p}_{\mathbf{t}}; x) d\nu(x) \epsilon_{\mathbf{t}}$$

In the case where  $\beta = 4$ , we have

$$\text{Wr}(p_{\mathbf{t}}; x) = p_{\mathbf{t}(1)}(x)p'_{\mathbf{t}(2)}(x) - p'_{\mathbf{t}(1)}(x)p_{\mathbf{t}(2)}(x); \quad \mathbf{t}: \underline{2} \nearrow \underline{2N},$$

and

$$\int_{\mathbb{R}} \omega(x) d\nu(x) = \sum_{\mathbf{t}: \underline{2} \nearrow \underline{2N}} \left\{ \int_{\mathbb{R}} [p_{\mathbf{t}(1)}(x)p'_{\mathbf{t}(2)}(x) - p'_{\mathbf{t}(1)}(x)p_{\mathbf{t}(2)}(x)] d\nu(x) \right\} \epsilon_{\mathbf{t}(1)} \wedge \epsilon_{\mathbf{t}(2)}.$$

This is exactly the 2-form associated to the antisymmetric matrix

$$\mathbf{W} = \left[ \int_{\mathbb{R}} [p_m(x)p'_n(x) - p'_m(x)p_n(x)] d\nu(x) \right]_{m,n=1}^{2N},$$

and  $Z_N = \text{Pf } \mathbf{W}$ , as is well known [7, 16].

When  $\beta = L^2$  is odd and  $N$  is even,

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \omega(x) \wedge \omega(y) \text{sgn}(y-x) d\nu(x) d\nu(y) \\ &= \sum_{\mathbf{t}, \mathbf{u}: \underline{L} \nearrow \underline{NL}} \left\{ \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \text{Wr}(\mathbf{p}_{\mathbf{t}}; x) \text{Wr}(\mathbf{p}_{\mathbf{u}}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right\} \epsilon_{\mathbf{t}} \wedge \epsilon_{\mathbf{u}} \end{aligned}$$

In the case where  $L = \beta = 1$ , each  $\mathbf{t} : \underline{1} \nearrow \underline{N}$  selects a single integer between 1 and  $N$ , and  $\text{Wr}(\mathbf{p}; x) = p_{\mathbf{t}(1)}(x)$ . It follows that

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \omega(x) \wedge \omega(y) \operatorname{sgn}(y - x) d\nu(x) d\nu(y) \\ &= \sum_{m,n=1}^N \left\{ \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} p_m(x) p_n(y) \operatorname{sgn}(y - x) d\nu(x) d\nu(y) \right\} \mathbf{e}_m \wedge \mathbf{e}_n \end{aligned}$$

This is the 2-form associated to the antisymmetric matrix

$$\mathbf{U} = \left[ \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} p_n(x) p_m(y) \operatorname{sgn}(y - x) d\nu(x) d\nu(y) \right]_{n,m=1}^N,$$

and deBruijn's identities in this case have that  $Z_N = \text{Pf } \mathbf{U}$ .

When  $L = 1$  and  $N$  is odd, the additional term on the right hand side of (3) correspond to bordering the matrix  $\mathbf{U}$  to produce

$$\mathbf{U}' = \begin{bmatrix} & & & \int_{\mathbb{R}} p_1(x) d\nu(x) \\ & & & \vdots \\ & \mathbf{U} & & \int_{\mathbb{R}} p_N(x) d\nu(x) \\ -\int_{\mathbb{R}} p_1(x) d\nu(x) & \cdots & -\int_{\mathbb{R}} p_N(x) d\nu(x) & 0 \end{bmatrix},$$

and in this situation,  $Z_N = \text{Pf } \mathbf{U}'$  as is well known [1].

The case where  $\beta = L^2 + 1$  is even and  $N = 2M$  does not seem to generalize any known situation, though it is apparently applicable when  $\beta = 2$ . The partition functions of  $\beta = 2$  ensembles are traditionally described in terms of determinants. And, while every determinant can be written trivially as a Pfaffian, the Pfaffian expression which appears for the partition function here does not seem to arise from such a trivial modification. At any rate, when  $\beta = 2$  we have that

$$Z_{2M} = \text{Pf } \mathbf{Y},$$

where

$$\mathbf{Y} = \left[ \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} p_m(x) p_n(y) \left( \frac{(y^M - x^M)^2}{y - x} \right) d\nu(x) d\nu(y) \right]_{m,n=1}^{2M}. \quad (2.2)$$

## 2.1 For Circular Ensembles

The proof of Theorem 2.1 is mostly formal, and similar results can be had for other ensembles. For instance, the joint density of Dyson's circular ensembles [8] is given by

$$Q_N(\boldsymbol{\lambda}) = \frac{1}{C_N N!} \prod_{m < n} |e^{i\theta_n} - e^{i\theta_m}|^\beta; \quad \theta_1, \theta_2, \dots, \theta_N \in [-\pi, \pi)$$

where

$$C_N = \frac{1}{N!} \int_{[-\pi, \pi]^N} \prod_{m < n} |e^{i\theta_n} - e^{i\theta_m}|^\beta d\theta_1 d\theta_2 \cdots d\theta_N.$$

By [16, (11.3.2)],

$$|e^{i\theta_n} - e^{i\theta_m}| = -ie^{-i(\theta_n + \theta_m)/2} \operatorname{sgn}(\theta_n - \theta_m)(e^{i\theta_n} - e^{i\theta_m}),$$

and thus, if we define,

$$d\mu(\theta) = (-ie^{-i\theta})^{(N-1)/2} d\theta, \quad (2.3)$$

then

$$C_N = \frac{1}{N!} \int_{[-\pi, \pi]^N} \left\{ \prod_{m < n} (e^{i\theta_n} - e^{i\theta_m}) \operatorname{sgn}(\theta_n - \theta_m) \right\}^\beta d\mu^N(\theta).$$

Of course the  $\operatorname{sgn}(\theta_n - \theta_m)$  terms can be ignored when  $\beta$  is even.

An explicit evaluation of  $C_N$  in the case where  $d\mu(\theta) = d\theta/2\pi$ , conjectured in [8] and proved by various means in [11, 10, 2, 23].

**Theorem 2.2.** *Suppose  $L$  and  $N$  are positive integers with  $\beta = L^2$  and let  $\omega(x)$  be the  $L$ -form given as in (2.1) and suppose  $\mu$  is given as in (2.3). Then, for any complete family of monic polynomials  $\mathbf{p}$ ,*

1. if  $\beta = L^2$

$$C_N = \operatorname{PF} \left( \int_{-\pi}^{\pi} \omega(e^{i\theta}) d\mu(\theta) \right);$$

2. if  $\beta = L^2$  is odd and  $N$  is even

$$C_N = \operatorname{PF} \left( \frac{1}{2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \omega(e^{i\theta}) \wedge \omega(e^{i\psi}) \operatorname{sgn}(\psi - \theta) d\mu(\theta) d\mu(\psi) \right);$$

3. if  $\beta = L^2$  is odd and  $N$  is odd

$$C_N = \operatorname{PF} \left( \frac{1}{2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \omega(e^{i\theta}) \wedge \omega(e^{i\psi}) \operatorname{sgn}(\psi - \theta) d\mu(\theta) d\mu(\psi) + \int_{-\pi}^{\pi} \omega(e^{i\theta}) \wedge \epsilon' d\mu(\theta) \right);$$

4. if  $\beta = L^2 + 1$  is even and  $N = 2M$  is even,

$$C_N = \operatorname{PF} \left( \frac{1}{2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \omega(e^{i\theta}) \wedge \omega(e^{i\psi}) \left[ \frac{(e^{iM\psi} - e^{iM\theta})^2}{e^{i\psi} - e^{i\theta}} \right] d\mu(\theta) d\mu(\psi) \right).$$

The proof of this theorem is the same, *mutadis mutandis*, as that of Theorem 2.1.

## 2.2 In Terms of Moments

For  $j, k \geq 0$ , let the  $k$ th moment of  $\nu$  be given by

$$M(k) = \int_{\mathbb{R}} x^k d\nu(x),$$

and the  $j, k$ th skew-moment of  $\nu$  to be

$$M(j, k) = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} x^j y^k \operatorname{sgn}(y - x) d\nu(x) d\nu(y).$$

If we set  $\mathbf{p}$  to be just the monomials, and define

$$\Delta \mathbf{t} = \prod_{j < k} (\mathbf{t}(k) - \mathbf{t}(j)) \quad \text{and} \quad \Sigma \mathbf{t} = \sum_{\ell=1}^L \mathbf{t}(\ell),$$

then the Wronskian of  $\mathbf{p}_t$  is given by [6]

$$\text{Wr}(\mathbf{p}_t; x) = \Delta t \left\{ \prod_{\ell=1}^L \frac{1}{(t(\ell) - 1)!} \right\} x^{\Sigma t - L(L+1)/2}.$$

It follows that

$$\begin{aligned} \omega(x) &= \sum_{t: \underline{L} \nearrow \underline{NL}} \Delta t \left\{ \prod_{\ell=1}^L \frac{1}{(t(\ell) - 1)!} \right\} x^{\Sigma t - L(L+1)/2} \epsilon_t, \\ \int_{\mathbb{R}} \omega(x) d\nu(x) &= \sum_{t: \underline{L} \nearrow \underline{NL}} \Delta t \left\{ \prod_{\ell=1}^L \frac{1}{(t(\ell) - 1)!} \right\} M\left(\Sigma t - \frac{L(L+1)}{2}\right) \epsilon_t \end{aligned}$$

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^2} \omega(x) \omega(y) \text{sgn}(y - x) d\nu(x) d\nu(y) \\ &= \sum_{t, u} \Delta t \Delta u \left\{ \prod_{\ell=1}^L \frac{1}{(t(\ell) - 1)! (u(\ell) - 1)!} \right\} M\left(\Sigma t - \frac{L(L+1)}{2}, \Sigma u - \frac{L(L+1)}{2}\right) \epsilon_t \wedge \epsilon_u. \end{aligned}$$

Similarly,

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^2} \omega(x) \omega(y) \text{sgn}(y - x) d\nu(x) d\nu(y) + \int_{\mathbb{R}} \omega(x) \wedge \epsilon' d\nu(x) \\ &= \sum_{t, u} \Delta t \Delta u \left\{ \prod_{\ell=1}^L \frac{1}{(t(\ell) - 1)! (u(\ell) - 1)!} \right\} M\left(\Sigma t - \frac{L(L+1)}{2}, \Sigma u - \frac{L(L+1)}{2}\right) \epsilon_t \wedge \epsilon_u \\ &\quad + \sum_{t: \underline{L} \nearrow \underline{NL}} \Delta t \left\{ \prod_{\ell=1}^L \frac{1}{(t(\ell) - 1)!} \right\} M\left(\Sigma t - \frac{L(L+1)}{2}\right) \epsilon_t \wedge \epsilon'. \end{aligned}$$

### 3 A Remark on Correlation Functions

We consider the joint density  $\Omega_N(\lambda)$  given as in (0.1), though the results here easily extend to circular and other related ensembles. We will define the measure  $d\mu(\lambda) = w(\lambda) d\lambda$  and define the  $n$ th correlation function is given by

$$R_n(x_1, x_2, \dots, x_n) = \frac{1}{Z_N^\mu (N - n)!} \int_{\mathbb{R}^{N-n}} \Omega_N(x_1, \dots, x_n, y_1, \dots, y_{N-n}) dy_1 \cdots dy_{N-n}.$$

It shall be convenient to abbreviate things so that

$$\Delta(\lambda) = \prod_{m < n} (\lambda_n - \lambda_m); \quad \lambda \in \mathbb{R}^N,$$

and if  $\mathbf{x} \in \mathbb{R}^n$  and  $\mathbf{y} \in \mathbb{R}^{N-n}$  then

$$\mathbf{x} \vee \mathbf{y} = (x_1, \dots, x_n, y_1, \dots, y_{N-n}).$$

Thus,

$$R_n(\mathbf{x}) = \frac{1}{Z_N^\mu(N-n)!} \int_{\mathbb{R}^{n-n}} |\Delta(\mathbf{x} \vee \mathbf{y})|^\beta d\mu^{N-n}(\mathbf{y}).$$

The  $n$ th correlation function is clearly a renormalized version of the  $n$ th marginal density of  $\Omega_N$ . However, this observation belies the importance of correlation functions in random matrix theory, point processes and statistical physics (see for instance [12, 4]).

Given indeterminants  $c_1, c_2, \dots, c_N$  and real numbers  $x_1, x_2, \dots, x_N$  we define the measure  $\nu$  on  $\mathbb{R}$  by

$$d\nu(\lambda) = w(\lambda) \sum_{n=1}^N c_n d\delta(\lambda - x_n),$$

where  $\delta$  is the probability measure with unit mass at 0. It is then a straightforward exercise in symbolic manipulation to show that  $Z_N^{\mu+\nu}/Z_N^\mu$  is the generating function for the correlation functions. That is,  $R_N(x_1, \dots, x_n)$  is the coefficient of  $c_1 c_2 \dots c_n$  in  $Z_N^{\mu+\nu}/Z_N^\mu$ .

When  $\beta = 1$  or  $4$ , this observation and the fact that  $Z_N^{\mu+\nu}$  is the first step in showing that the correlation functions have a Pfaffian formulation. The simplest derivation is given by Tracy and Widom [22] using the fact that if  $\mathbf{A}$  and  $\mathbf{B}$  are (perhaps rectangular) matrices for which both  $\mathbf{AB}$  and  $\mathbf{BA}$  are square, then

$$\det(\mathbf{I} + \mathbf{AB}) = \det(\mathbf{I} + \mathbf{BA}). \quad (3.1)$$

A similar fact is true for Pfaffians [18] which streamlines the proof [5, 21].

The existence of a hyperpfaffian representation of  $Z_N^{\mu+\nu}$  when  $\beta$  is a square is suggestive of a hyperpfaffian formulation of the correlation functions, however the necessary analog of (3.1) for hyperpfaffians remains unknown.

## 4 The Proof of Theorem 2.1

### 4.1 Case: $\beta = L^2$ even

We define the confluent  $NL \times NL$  Vandermonde matrix by first defining the  $NL \times L$  matrix

$$\mathbf{V}(x) = [D^{\ell-1} p_n(x)]_{n,\ell=1}^{NL,L},$$

and then defining

$$\mathbf{V}(\boldsymbol{\lambda}) = [\mathbf{V}(\lambda_1) \quad \mathbf{V}(\lambda_2) \quad \dots \quad \mathbf{V}(\lambda_N)].$$

The confluent Vandermonde identity has that

$$\det \mathbf{V}(\boldsymbol{\lambda}) = \prod_{m < n} (\lambda_n - \lambda_m)^{L^2} = \prod_{m < n} |\lambda_n - \lambda_m|^\beta.$$

It follows that

$$Z_N = \frac{1}{N!} \int_{\mathbb{R}^N} \det \mathbf{V}(\boldsymbol{\lambda}) d\nu^N(\boldsymbol{\lambda}).$$

We may use Laplace expansion to write  $\det \mathbf{V}(\boldsymbol{\lambda})$  as a sum of  $N$ -fold products of  $L \times L$  determinants, where each determinant appearing in the products is a function of exactly



one of the variables  $\lambda_1, \lambda_2, \dots, \lambda_N$ . That is, given  $\mathbf{t} : \underline{L} \nearrow \underline{NL}$ , define  $\mathbf{V}_{\mathbf{t}}(\lambda)$  to be the  $L \times L$  matrix,

$$\mathbf{V}_{\mathbf{t}}(\lambda) = [D^{\ell-1} p_{\mathbf{t}(n)}(\lambda)]_{n, \ell=1}^L.$$

Then, we may write  $\det \mathbf{V}(\lambda)$  as an alternating sum over products of the form

$$\det \mathbf{V}_{\mathbf{t}_1}(\lambda_1) \cdot \det \mathbf{V}_{\mathbf{t}_2}(\lambda_2) \cdot \dots \cdot \det \mathbf{V}_{\mathbf{t}_N}(\lambda_N),$$

where  $\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N : \underline{L} \nearrow \underline{NL}$  have ranges which are mutually disjoint (or equivalent, the disjoint union of their ranges is all of  $\underline{NL}$ ). The sign of each term in the sum can be specified by defining  $\text{sgn}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N)$  via

$$\epsilon_{\mathbf{t}_1} \wedge \epsilon_{\mathbf{t}_2} \wedge \dots \wedge \epsilon_{\mathbf{t}_N} = \text{sgn}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N) \cdot \epsilon_{\text{Vol}}.$$

We remark that, if  $\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N : \underline{L} \nearrow \underline{NL}$  do not have disjoint ranges, then  $\text{sgn}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N) = 0$ . Thus we can write

$$\det \mathbf{V}(\lambda) = \sum_{(\mathbf{t}_1, \dots, \mathbf{t}_N)} \text{sgn}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N) \det \mathbf{V}_{\mathbf{t}_1}(\lambda_1) \cdot \det \mathbf{V}_{\mathbf{t}_2}(\lambda_2) \cdot \dots \cdot \det \mathbf{V}_{\mathbf{t}_N}(\lambda_N), \quad (4.1)$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} \det \mathbf{V}(\lambda) d\nu^N(\lambda) &= \sum_{(\mathbf{t}_1, \dots, \mathbf{t}_N)} \text{sgn}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N) \prod_{n=1}^N \int_{\mathbb{R}} \det \mathbf{V}_{\mathbf{t}_n}(\lambda) d\nu(\lambda) \\ &= \sum_{(\mathbf{t}_1, \dots, \mathbf{t}_N)} \text{sgn}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N) \prod_{n=1}^N \int_{\mathbb{R}} \text{Wr}(\mathbf{p}_{\mathbf{t}_n}; x) d\nu(x) \end{aligned}$$

It follows that

$$\begin{aligned} \frac{1}{N!} \left\{ \int_{\mathbb{R}^N} \det \mathbf{V}(\lambda) d\nu^N(\lambda) \right\}_{\epsilon_{\text{Vol}}} &= \frac{1}{N!} \sum_{(\mathbf{t}_1, \dots, \mathbf{t}_N)} \left\{ \prod_{n=1}^N \int_{\mathbb{R}} \text{Wr}(\mathbf{p}_{\mathbf{t}_n}; x) d\nu(x) \right\} \text{sgn}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N) \epsilon_{\text{Vol}} \\ &= \frac{1}{N!} \sum_{(\mathbf{t}_1, \dots, \mathbf{t}_N)} \left\{ \bigwedge_{n=1}^N \int_{\mathbb{R}} \text{Wr}(\mathbf{p}_{\mathbf{t}_n}; x) d\nu(x) \epsilon_{\mathbf{t}_n} \right\}. \end{aligned}$$

Finally, exchanging the sum and the wedge product, we find

$$\frac{1}{N!} \left\{ \int_{\mathbb{R}^N} \det \mathbf{V}(\lambda) d\nu^N(\lambda) \right\}_{\epsilon_{\text{Vol}}} = \frac{1}{N!} \left\{ \sum_{\mathbf{t} : \underline{N} \nearrow \underline{NL}} \int_{\mathbb{R}} \text{Wr}(\mathbf{p}_{\mathbf{t}}; x) d\nu(x) \epsilon_{\mathbf{t}} \right\}^{\wedge N},$$

which proves this case of the theorem.

## 4.2 Case: $\beta = L^2$ odd, $N$ even

When  $L$  is odd, the situation is complicated by the fact that

$$\prod_{m < n} |\lambda_n - \lambda_m|^\beta = |\det \mathbf{V}(\lambda)|.$$

(Note the absolute values). Thus, in this case

$$Z_N = \frac{1}{N!} \int_{\mathbb{R}^N} |\det \mathbf{V}(\boldsymbol{\lambda})| d\nu^N(\boldsymbol{\lambda}).$$

This complication is eased by the observation, [7, Eq. 5.3], that if

$$\mathbf{T}(\boldsymbol{\lambda}) = [\text{sgn}(\lambda_n - \lambda_m)]_{m,n=1}^N, \quad \text{then} \quad \text{Pf } \mathbf{T}(\boldsymbol{\lambda}) = \prod_{m < n} \text{sgn}(\lambda_n - \lambda_m), \quad (4.2)$$

and thus

$$Z_N = \frac{1}{N!} \int_{\mathbb{R}^N} \det \mathbf{V}(\boldsymbol{\lambda}) \text{Pf } \mathbf{T}(\boldsymbol{\lambda}) d\nu^N(\boldsymbol{\lambda}).$$

We remark, it is at this point that we require  $N$  be even, since the Pfaffian of  $\mathbf{T}(\boldsymbol{\lambda})$  is not defined when  $N$  is odd.

Using (4.1), we have

$$Z_N = \frac{1}{N!} \sum_{\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N} \text{sgn}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N) \int_{\mathbb{R}^N} \left\{ \prod_{n=1}^N \det \mathbf{V}_{\mathbf{t}_n}(\boldsymbol{\lambda}_n) \right\} \text{Pf } \mathbf{T}(\boldsymbol{\lambda}) d\nu^N(\boldsymbol{\lambda}),$$

where as before the sum is over  $N$ -tuples of functions  $\underline{L} \nearrow \underline{NL}$ . Now, if  $N = 2M$ , then

$$\text{Pf } \mathbf{T}(\boldsymbol{\lambda}) = \frac{1}{M!2^M} \sum_{\sigma \in S_N} \text{sgn } \sigma \prod_{m=1}^M \text{sgn}(\lambda_{\sigma(2m)} - \lambda_{\sigma(2m-1)}),$$

and

$$\prod_{n=1}^N \det \mathbf{V}_{\mathbf{t}_n}(\boldsymbol{\lambda}_n) = \prod_{m=1}^M \det \mathbf{V}_{\mathbf{t}_{\sigma(2m-1)}}(\lambda_{\sigma(2m-1)}) \det \mathbf{V}_{\mathbf{t}_{\sigma(2m)}}(\lambda_{\sigma(2m)}).$$

Thus,

$$\begin{aligned} Z_N &= \frac{1}{N!M!2^M} \sum_{\sigma \in S_N} \sum_{\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N} \text{sgn } \sigma \text{sgn}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N) \\ &\quad \times \int_{\mathbb{R}^N} \left\{ \prod_{m=1}^M \det \mathbf{V}_{\mathbf{t}_{\sigma(2m-1)}}(\lambda_{\sigma(2m-1)}) \det \mathbf{V}_{\mathbf{t}_{\sigma(2m)}}(\lambda_{\sigma(2m)}) \right. \\ &\quad \left. \times \text{sgn}(\lambda_{\sigma(2m)} - \lambda_{\sigma(2m-1)}) \right\} d\nu^N(\boldsymbol{\lambda}) \\ &= \frac{1}{N!M!} \sum_{\sigma \in S_N} \sum_{\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N} \text{sgn } \sigma \text{sgn}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N) \\ &\quad \times \left\{ \prod_{m=1}^M \frac{1}{2} \int_{\mathbb{R}^2} \det \mathbf{V}_{\mathbf{t}_{\sigma(2m-1)}}(x) \det \mathbf{V}_{\mathbf{t}_{\sigma(2m)}}(y) \text{sgn}(y - x) d\nu(x) d\nu(y) \right\}. \end{aligned}$$

Next, we notice that  $S_N$  acts on the  $N$ -tuples  $(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N)$  by permutation, and if

$$(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N) = (\mathbf{t}_{\sigma(1)}, \mathbf{t}_{\sigma(2)}, \dots, \mathbf{t}_{\sigma(N)}),$$

then

$$\text{sgn}(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N) = \text{sgn}(\sigma) \text{sgn}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N).$$

Moreover, the action of each  $\sigma$  produces a bijection on the set of  $N$ -tuples, and thus

$$Z_N = \frac{1}{N!M!} \sum_{\sigma \in S_N} \sum_{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N} \text{sgn}(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N) \\ \times \left\{ \prod_{m=1}^M \int_{\mathbb{R}^2} \frac{1}{2} \det \mathbf{V}_{\mathbf{u}_{2m-1}}(x) \det \mathbf{V}_{\mathbf{u}_{2m}}(y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right\}.$$

The summand is independent of  $\sigma$ , and thus

$$Z_N = \frac{1}{M!} \sum_{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N} \text{sgn}(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N) \\ \times \left\{ \prod_{m=1}^M \frac{1}{2} \int_{\mathbb{R}^2} \det \mathbf{V}_{\mathbf{u}_{2m-1}}(x) \det \mathbf{V}_{\mathbf{u}_{2m}}(y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right\}.$$

Now, every  $N$ -tuple  $(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N)$  can be obtained from an  $M$ -tuple  $(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_M)$  of functions  $\underline{2L} \nearrow \underline{2LM}$  and an  $M$ -tuple  $(\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_M)$  of functions  $\underline{L} \nearrow \underline{2L}$  specified by

$$\mathbf{u}_{2m-1} = \mathbf{v}_m \circ \mathbf{w}_m \quad \text{and} \quad \mathbf{u}_{2m} = \mathbf{v}_m \circ \mathbf{w}'_m,$$

where  $\mathbf{w}'_m$  is the unique function  $\underline{L} \nearrow \underline{2L}$  whose range is disjoint from  $\mathbf{w}_m$ . Defining the sign of each of the  $\mathbf{w}_m$  by

$$\epsilon_{\mathbf{w}_m} \wedge \epsilon_{\mathbf{w}'_m} = \text{sgn } \mathbf{w}_m (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \dots \wedge \mathbf{e}_{2L}),$$

we have

$$\text{sgn}(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N) = \text{sgn}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_M) \prod_{m=1}^M \text{sgn } \mathbf{w}_m.$$

Using this decomposition,

$$Z_N = \frac{1}{M!} \sum_{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_M} \text{sgn}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_M) \\ \times \sum_{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_M} \left\{ \prod_{m=1}^M \frac{\text{sgn } \mathbf{w}_m}{2} \int_{\mathbb{R}^2} \det \mathbf{V}_{\mathbf{v}_m \circ \mathbf{w}_m}(x) \det \mathbf{V}_{\mathbf{v}_m \circ \mathbf{w}'_m}(y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right\} \\ = \frac{1}{M!} \sum_{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_M} \text{sgn}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_M) \\ \times \left\{ \prod_{m=1}^M \left[ \frac{1}{2} \sum_{\mathbf{w}: \underline{L} \nearrow \underline{2L}} \text{sgn } \mathbf{w} \int_{\mathbb{R}^2} \det \mathbf{V}_{\mathbf{v}_m \circ \mathbf{w}}(x) \det \mathbf{V}_{\mathbf{v}_m \circ \mathbf{w}'}(y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right] \right\},$$

and consequently,

$$\begin{aligned}
Z_N \epsilon_{\text{vol}} &= \frac{1}{M!} \sum_{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_M} \left\{ \bigwedge_{m=1}^M \left[ \frac{1}{2} \sum_{\mathbf{w}: \underline{L} \nearrow \underline{2L}} \text{sgn } \mathbf{w} \int_{\mathbb{R}^2} \det \mathbf{V}_{\mathbf{v}_m \circ \mathbf{w}}(x) \det \mathbf{V}_{\mathbf{v}_m \circ \mathbf{w}'}(y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right] \epsilon_{\mathbf{v}_m} \right\} \\
&= \frac{1}{M!} \left( \frac{1}{2} \sum_{\mathbf{v}: \underline{2L} \nearrow \underline{2LM}} \left[ \sum_{\mathbf{w}: \underline{L} \nearrow \underline{2L}} \text{sgn } \mathbf{w} \int_{\mathbb{R}^2} \det \mathbf{V}_{\mathbf{v} \circ \mathbf{w}}(x) \det \mathbf{V}_{\mathbf{v} \circ \mathbf{w}'}(y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right] \epsilon_{\mathbf{v}} \right)^{\wedge M}.
\end{aligned}$$

Thus, substituting  $\text{Wr}(\mathbf{p}_{\mathbf{v} \circ \mathbf{w}}; x) = \det \mathbf{V}_{\mathbf{v} \circ \mathbf{w}}(x)$  (and likewise for  $\text{Wr}(\mathbf{p}_{\mathbf{v} \circ \mathbf{w}'}; y)$ ) we have  $Z_N$  equal to

$$\text{PF} \left( \frac{1}{2} \sum_{\mathbf{v}: \underline{2L} \nearrow \underline{2LM}} \left[ \sum_{\mathbf{w}: \underline{L} \nearrow \underline{2L}} \text{sgn } \mathbf{w} \int_{\mathbb{R}^2} \text{Wr}(\mathbf{p}_{\mathbf{v} \circ \mathbf{w}}; x) \text{Wr}(\mathbf{p}_{\mathbf{v} \circ \mathbf{w}'}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right] \epsilon_{\mathbf{v}} \right).$$

Next, we use the fact that  $\epsilon_{\mathbf{v}} = \text{sgn } \mathbf{w} \epsilon_{\mathbf{v} \circ \mathbf{w}} \wedge \epsilon_{\mathbf{v} \circ \mathbf{w}'}$  to write

$$\begin{aligned}
&\frac{1}{2} \sum_{\mathbf{v}: \underline{2L} \nearrow \underline{2LM}} \left[ \sum_{\mathbf{w}: \underline{L} \nearrow \underline{2L}} \text{sgn } \mathbf{w} \int_{\mathbb{R}^2} \text{Wr}(\mathbf{p}_{\mathbf{v} \circ \mathbf{w}}; x) \text{Wr}(\mathbf{p}_{\mathbf{v} \circ \mathbf{w}'}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right] \epsilon_{\mathbf{v}} \\
&= \frac{1}{2} \sum_{\mathbf{v}: \underline{2L} \nearrow \underline{2LM}} \sum_{\mathbf{w}: \underline{L} \nearrow \underline{2L}} \left\{ \int_{\mathbb{R}^2} \text{Wr}(\mathbf{p}_{\mathbf{v} \circ \mathbf{w}}; x) \text{Wr}(\mathbf{p}_{\mathbf{v} \circ \mathbf{w}'}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right\} \epsilon_{\mathbf{v} \circ \mathbf{w}} \wedge \epsilon_{\mathbf{v} \circ \mathbf{w}'}.
\end{aligned} \tag{4.3}$$

Finally, we see that as  $\mathbf{v}$  varies over  $\underline{2L} \nearrow \underline{2LM}$  and  $\mathbf{w}$  varies over  $\underline{L} \nearrow \underline{2L}$ ,  $\mathbf{t} = \mathbf{v} \circ \mathbf{w}$  and  $\mathbf{u} = \mathbf{v} \circ \mathbf{w}'$  vary over pairs in  $\underline{L} \nearrow \underline{2LM}$  with disjoint ranges. However, since  $\epsilon_{\mathbf{t}} \wedge \epsilon_{\mathbf{u}} = 0$  if the ranges of  $\mathbf{t}$  and  $\mathbf{u}$  are not disjoint, we may replace the double sum in (4.3) with a double sum over  $\underline{L} \nearrow \underline{2LM}$ . That is, (4.3) equals

$$\begin{aligned}
&\frac{1}{2} \sum_{\mathbf{t}: \underline{L} \nearrow \underline{2LM}} \sum_{\mathbf{u}: \underline{L} \nearrow \underline{2LM}} \left\{ \int_{\mathbb{R}^2} \text{Wr}(\mathbf{p}_{\mathbf{t}}; x) \text{Wr}(\mathbf{p}_{\mathbf{u}}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right\} \epsilon_{\mathbf{t}} \wedge \epsilon_{\mathbf{u}} \\
&= \frac{1}{2} \int_{\mathbb{R}^2} \left\{ \sum_{\mathbf{t}: \underline{L} \nearrow \underline{2LM}} \sum_{\mathbf{u}: \underline{L} \nearrow \underline{2LM}} \text{Wr}(\mathbf{p}_{\mathbf{t}}; x) \text{Wr}(\mathbf{p}_{\mathbf{u}}; y) \text{sgn}(y-x) \epsilon_{\mathbf{t}} \wedge \epsilon_{\mathbf{u}} \right\} d\nu(x) d\nu(y) \\
&= \frac{1}{2} \int_{\mathbb{R}^2} \omega(x) \wedge \omega(y) \text{sgn}(y-x) d\nu(x) d\nu(y),
\end{aligned}$$

as desired.

### 4.3 Case: $\beta = L^2$ odd, $N$ odd

In this section we will assume that  $N = 2K - 1$  is odd and for  $\lambda \in \mathbb{R}^N$  we will introduce the  $2K \times 2K$  matrix  $\mathbf{T}'(\lambda)$  by

$$\mathbf{T}'(\lambda) = [t_{m,n}] = \begin{bmatrix} & & & 1 \\ & \mathbf{T}(\lambda) & & \vdots \\ & & & 1 \\ -1 & \cdots & -1 & 0 \end{bmatrix}.$$

That is

$$\mathbf{T}'(\lambda) = \mathbf{T}(\lambda \vee \infty) = \lim_{t \rightarrow \infty} \mathbf{T}(\lambda \vee t),$$

where, for instance,  $(\lambda \vee t) = (\lambda_1, \dots, \lambda_N, t) \in \mathbb{R}^{N+1}$ . From (4.2),

$$\text{PF } \mathbf{T}'(\lambda) = \prod_{1 \leq m < n \leq N} \text{sgn}(\lambda_n - \lambda_m).$$

Here we will write

$$\Pi_{2K} = \{\sigma \in S_{2K} : \sigma(2k-1) < \sigma(2k) \text{ for } k = 1, 2, \dots, K\},$$

and expand the Pfaffian of  $\mathbf{U}(\lambda)$  as

$$\text{PF } \mathbf{U}(\lambda) = \frac{1}{K!} \sum_{\sigma \in \Pi_{2K}} \text{sgn } \sigma \prod_{k=1}^K t_{\sigma(2k-1), \sigma(2k)}.$$

For each  $\sigma \in \Pi_{2K}$  there exists  $k_\sigma \in \underline{K}$  such that  $\sigma(2k_\sigma) = 2K$ , and hence,  $t_{\sigma(2k_\sigma-1), \sigma(2k_\sigma)} = 1$ . That is,

$$\text{PF } \mathbf{U}(\lambda) = \frac{1}{K!} \sum_{\sigma \in \Pi_{2K}} \text{sgn } \sigma \prod_{\substack{k=1 \\ k \neq k_\sigma}}^K \text{sgn}(\lambda_{\sigma(2k)} - \lambda_{\sigma(2k-1)}).$$

Using this, and following the outline of the  $L$  odd  $N$  even case, we find

$$\begin{aligned} Z_N &= \frac{1}{N!K!} \sum_{\sigma \in \Pi_{2K}} \sum_{\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N} \text{sgn } \sigma \text{sgn}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N) \int_{\mathbb{R}^N} \text{Wr}(\mathbf{p}_{\mathbf{t}_{\sigma(2k_\sigma-1)}}; \lambda_{\sigma(2k_\sigma-1)}) \\ &\quad \times \left\{ \prod_{\substack{k=1 \\ k \neq k_\sigma}}^K \text{Wr}(\mathbf{p}_{\mathbf{t}_{\sigma(2k-1)}}; \lambda_{\sigma(2k-1)}) \text{Wr}(\mathbf{p}_{\mathbf{t}_{\sigma(2k)}}; \lambda_{\sigma(2k)}) \text{sgn}(\lambda_{\sigma(2k)} - \lambda_{\sigma(2k-1)}) \right\} d\nu^N(\lambda), \end{aligned}$$

where, as before,  $(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N)$  is an  $N$ -tuple of functions  $\underline{L} \nearrow \underline{NL}$ . Fubini's Theorem then yields

$$\begin{aligned} Z_N &= \frac{1}{N!K!} \sum_{\sigma \in \Pi_{2K}} \sum_{\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N} \text{sgn } \sigma \text{sgn}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N) \int_{\mathbb{R}} \text{Wr}(\mathbf{p}_{\mathbf{t}_{\sigma(2k_\sigma-1)}}; x) d\nu(x) \\ &\quad \times \left\{ \prod_{\substack{k=1 \\ k \neq k_\sigma}}^K \int_{\mathbb{R}^2} \text{Wr}(\mathbf{p}_{\mathbf{t}_{\sigma(2k-1)}}; x) \text{Wr}(\mathbf{p}_{\mathbf{t}_{\sigma(2k)}}; y) \text{sgn}(y - x) d\nu(x) d\nu(y) \right\} \end{aligned}$$

Now, given  $\sigma \in \Pi_{2K}$ , let  $\tilde{\sigma} \in \Pi_{2K}$  be the permutation given by

$$\tilde{\sigma}(n) = \begin{cases} \sigma(n) & \text{if } n \leq 2k_\sigma; \\ \sigma(n+2) & \text{if } 2k_\sigma + 2 \leq n \leq 2K-2; \\ \sigma(2k_\sigma - 1) & \text{if } n = 2K-1; \\ 2K & \text{if } n = 2K. \end{cases}$$

That is,  $\tilde{\sigma}$  is the permutation whose range as an ordered set is equal to that formed from the range of  $\sigma$  by ‘moving’  $\sigma(2k_\sigma - 1)$  and  $\sigma(2k_\sigma)$  to the ‘end.’ Clearly  $\text{sgn } \tilde{\sigma} = \text{sgn } \sigma$ , and each  $\tilde{\sigma}$  corresponds to exactly  $K$  distinct  $\sigma \in \Pi_{2K}$ . It follows that

$$Z_N = \frac{1}{N!K!} \sum_{\sigma \in \Pi_{2K}} \sum_{\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N} \text{sgn } \tilde{\sigma} \text{sgn}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N) \int_{\mathbb{R}} \text{Wr}(\mathbf{p}_{\mathbf{t}_{\tilde{\sigma}(2K-1)}}; x) d\nu(x) \\ \times \left\{ \prod_{k=1}^{K-1} \int_{\mathbb{R}^2} \text{Wr}(\mathbf{p}_{\mathbf{t}_{\tilde{\sigma}(2k-1)}}; x) \text{Wr}(\mathbf{p}_{\mathbf{t}_{\tilde{\sigma}(2k)}}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right\}$$

Next, denote the transpositions  $\tau_k = (2k-1, 2k)$  for  $k = 1, 2, \dots, K-1$  and let  $G_{2K-2}$  be the group of permutations generated by the  $\tau_k$ . Clearly the cardinality of  $G_{2K-2}$  is  $2^{K-1}$ , and moreover the map

$$\Pi_{2K} \times G_{2K-2} \rightarrow \{\pi \in S_{2K} : \pi(2K) = 2K\} \quad (4.4)$$

given by  $(\sigma, \tau) \mapsto \tilde{\sigma} \circ \tau$  is a  $K$  to one map. Clearly, the right hand set is in correspondence with  $S_N$ . Now, if  $\pi = \tilde{\sigma} \circ \tau$  for some  $\sigma \in \Pi_{2K}$  and  $\tau \in G_{2K-2}$  then,

$$\text{sgn } \tilde{\sigma} \left\{ \prod_{k=1}^{K-1} \int_{\mathbb{R}^2} \text{Wr}(\mathbf{p}_{\mathbf{t}_{\tilde{\sigma}(2k-1)}}; x) \text{Wr}(\mathbf{p}_{\mathbf{t}_{\tilde{\sigma}(2k)}}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right\} \int_{\mathbb{R}} \text{Wr}(\mathbf{p}_{\mathbf{t}_{\tilde{\sigma}(2K-1)}}; x) d\nu(x) \\ = \text{sgn } \pi \left\{ \prod_{k=1}^{K-1} \int_{\mathbb{R}^2} \text{Wr}(\mathbf{p}_{\mathbf{t}_{\pi(2k-1)}}; x) \text{Wr}(\mathbf{p}_{\mathbf{t}_{\pi(2k)}}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right\} \\ \times \int_{\mathbb{R}} \text{Wr}(\mathbf{p}_{\mathbf{t}_{\pi(2K-1)}}; x) d\nu(x),$$

the factors of  $-1$  introduced into  $\pi$  by the transpositions in  $\tau$  being compensated by the fact that the double integral swaps sign when the arguments are swapped. It follows that we may replace the sum over  $\Pi_{2K}$  with a sum over  $S_N$  so long as we compensate for the cardinality of  $G_{2K-2}$  and the fact that the map is  $K$  to 1. That is,

$$Z_N = \frac{1}{N!(K-1)!} \sum_{\pi \in S_N} \sum_{\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N} \text{sgn } \pi \text{sgn}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N) \\ \times \left\{ \prod_{k=1}^{K-1} \frac{1}{2} \int_{\mathbb{R}^2} \text{Wr}(\mathbf{p}_{\mathbf{t}_{\pi(2k-1)}}; x) \text{Wr}(\mathbf{p}_{\mathbf{t}_{\pi(2k)}}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right\} \\ \times \int_{\mathbb{R}} \text{Wr}(\mathbf{p}_{\mathbf{t}_{\pi(N)}}; x) d\nu(x).$$

As before

$$\operatorname{sgn} \pi \operatorname{sgn}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N) = \operatorname{sgn}(\mathbf{t}_{\pi(1)}, \mathbf{t}_{\pi(2)}, \dots, \mathbf{t}_{\pi(N)}),$$

and the action of any particular  $\pi \in S_N$  on the  $N$ -tuple  $(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N)$  is a bijection on the set of such  $N$ -tuples. Thus, we may eliminate the sum over  $S_N$  so long as we compensate by  $N!$ . That is,

$$\begin{aligned} Z_N &= \frac{1}{(K-1)!} \sum_{\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N} \operatorname{sgn}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N) \\ &\quad \times \left\{ \prod_{k=1}^{K-1} \frac{1}{2} \int_{\mathbb{R}^2} \operatorname{Wr}(\mathbf{p}_{\mathbf{t}_{2k-1}}; x) \operatorname{Wr}(\mathbf{p}_{\mathbf{t}_{2k}}; y) \operatorname{sgn}(y-x) d\nu(x) d\nu(y) \right\} \\ &\quad \times \int_{\mathbb{R}} \operatorname{Wr}(\mathbf{p}_{\mathbf{t}_N}; x) d\nu(x). \end{aligned} \quad (4.5)$$

With a slight modification of the argument in the  $L$  odd  $N$  even case, we set  $\mathbf{t} = \mathbf{t}_N$  and write  $(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t})$  as

$$(\mathbf{t}' \circ \mathbf{v}_1 \circ \mathbf{w}_1, \mathbf{t}' \circ \mathbf{v}_1 \circ \mathbf{w}'_1, \dots, \mathbf{t}' \circ \mathbf{v}_{K-1} \circ \mathbf{w}_{K-1}, \mathbf{t}' \circ \mathbf{v}_{K-1} \circ \mathbf{w}'_{K-1}, \mathbf{t}), \quad (4.6)$$

where  $(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{K-1})$  is a  $K-1$ -tuple of functions  $\underline{2L} \nearrow (N-1)L$  the union of ranges of which is all of  $\underline{L(N-1)}$ , and  $(\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_{K-1})$  is a  $K-1$ -tuple of functions  $\underline{L} \nearrow \underline{2L}$ . In words, (4.6) is formed by first forming a partition of  $(N-1)L$  into  $L$  disjoint subsets given by the ranges of the  $\mathbf{v}$ s. The  $\mathbf{w}$ s serve to divide each of these sets into two equal sized sets giving a partition of  $(N-1)L$  into  $2L$  disjoint subsets. Finally,  $\mathbf{t}'$  is the function from  $\underline{N(L-1)} \nearrow \underline{NL}$  whose range is disjoint from  $\mathbf{t}_N$ . Each  $(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_N)$  has a unique representation of this form, and

$$\operatorname{sgn}(\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_{N-1}, \mathbf{t}) = \operatorname{sgn} \mathbf{t}' \operatorname{sgn}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{K-1}) \prod_{k=1}^{K-1} \operatorname{sgn} \mathbf{w}_k.$$

With these observations, we may rewrite (4.5) as

$$\begin{aligned} Z_N &= \frac{1}{(K-1)!} \sum_{\mathbf{t}: \underline{L} \nearrow \underline{NL}} \left[ \operatorname{sgn} \mathbf{t}' \int_{\mathbb{R}} \operatorname{Wr}(\mathbf{p}_{\mathbf{t}}; x) d\nu(x) \sum_{\mathbf{v}_1, \dots, \mathbf{v}_{K-1}} \sum_{\mathbf{w}_1, \dots, \mathbf{w}_{K-1}} \operatorname{sgn}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{K-1}) \right. \\ &\quad \times \left. \left\{ \prod_{k=1}^{K-1} \frac{\operatorname{sgn} \mathbf{w}_k}{2} \int_{\mathbb{R}^2} \operatorname{Wr}(\mathbf{p}_{\mathbf{t}' \circ \mathbf{v}_k \circ \mathbf{w}_k}; x) \operatorname{Wr}(\mathbf{p}_{\mathbf{t}' \circ \mathbf{v}_k \circ \mathbf{w}'_k}; y) \operatorname{sgn}(y-x) d\nu(x) d\nu(y) \right\} \right], \end{aligned}$$

and thus, since  $\epsilon_{\text{vol}} = \operatorname{sgn} \mathbf{t}' \epsilon_{\mathbf{t}'} \wedge \epsilon_{\mathbf{t}}$ ,

$$\begin{aligned} Z_N \epsilon_{\text{vol}} &= \frac{1}{(K-1)!} \sum_{\mathbf{t}: \underline{L} \nearrow \underline{NL}} \left[ \int_{\mathbb{R}} \operatorname{Wr}(\mathbf{p}_{\mathbf{t}}; x) d\nu(x) \sum_{\mathbf{v}_1, \dots, \mathbf{v}_{K-1}} \sum_{\mathbf{w}_1, \dots, \mathbf{w}_{K-1}} \operatorname{sgn}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{K-1}) \right. \\ &\quad \times \left. \prod_{k=1}^{K-1} \left\{ \frac{\operatorname{sgn} \mathbf{w}_k}{2} \int_{\mathbb{R}^2} \operatorname{Wr}(\mathbf{p}_{\mathbf{t}' \circ \mathbf{v}_k \circ \mathbf{w}_k}; x) \operatorname{Wr}(\mathbf{p}_{\mathbf{t}' \circ \mathbf{v}_k \circ \mathbf{w}'_k}; y) \operatorname{sgn}(y-x) d\nu(x) d\nu(y) \right\} \epsilon_{\mathbf{t}'} \wedge \epsilon_{\mathbf{t}} \right]. \end{aligned}$$

Now,

$$\epsilon_{\mathbf{t}'} = \operatorname{sgn}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{K-1}) \left\{ \prod_{k=1}^{K-1} \operatorname{sgn} \mathbf{w}_k \right\} \bigwedge_{k=1}^{K-1} \epsilon_{\mathbf{t}' \circ \mathbf{v}_k \circ \mathbf{w}_k} \wedge \epsilon_{\mathbf{t}' \circ \mathbf{v}_k \circ \mathbf{w}'_k}$$

and thus,

$$Z_N \epsilon_{\text{vol}} = \frac{1}{(K-1)!} \sum_{\mathbf{t}; \underline{L} \nearrow \underline{NL}} \left[ \sum_{\mathbf{v}_1, \dots, \mathbf{v}_{K-1}} \sum_{\mathbf{w}_1, \dots, \mathbf{w}_{K-1}} \left\{ \bigwedge_{k=1}^{K-1} \frac{1}{2} \int_{\mathbb{R}^2} \text{Wr}(\mathbf{p}_{\mathbf{t}' \circ \mathbf{v}_k \circ \mathbf{w}_k}; x) \text{Wr}(\mathbf{p}_{\mathbf{t}' \circ \mathbf{v}_k \circ \mathbf{w}'_k}; y) \right. \right. \\ \left. \left. \times \text{sgn}(y-x) d\nu(x) d\nu(y) \epsilon_{\mathbf{t}' \circ \mathbf{v}_k \circ \mathbf{w}_k} \wedge \epsilon_{\mathbf{t}' \circ \mathbf{v}_k \circ \mathbf{w}'_k} \right\} \right. \\ \left. \wedge \int_{\mathbb{R}} \text{Wr}(\mathbf{p}_{\mathbf{t}}; x) d\nu(x) \epsilon_{\mathbf{t}} \right].$$

Using more-or-less the same maneuvers as in the  $L$  odd,  $N$  even case, we may exchange the sums and the  $K-1$ -fold wedge product to find

$$Z_N \epsilon_{\text{vol}} = \sum_{\mathbf{t}; \underline{L} \nearrow \underline{NL}} \frac{1}{(K-1)!} \\ \left( \sum_{\mathbf{s}, \mathbf{u}; \underline{L} \nearrow \underline{(N-1)L}} \frac{1}{2} \int_{\mathbb{R}^2} \text{Wr}(\mathbf{p}_{\mathbf{t}' \circ \mathbf{s}}; x) \text{Wr}(\mathbf{p}_{\mathbf{t}' \circ \mathbf{u}}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \epsilon_{\mathbf{t}' \circ \mathbf{s}} \wedge \epsilon_{\mathbf{t}' \circ \mathbf{u}} \right)^{\wedge(K-1)} \\ \wedge \int_{\mathbb{R}} \text{Wr}(\mathbf{p}_{\mathbf{t}}; x) d\nu(x) \epsilon_{\mathbf{t}},$$

and thus,

$$Z_N \epsilon'_{\text{vol}} = \sum_{\mathbf{t}; \underline{L} \nearrow \underline{NL}} \frac{1}{(K-1)!} \\ \left( \sum_{\mathbf{s}, \mathbf{u}; \underline{L} \nearrow \underline{(N-1)L}} \frac{1}{2} \int_{\mathbb{R}^2} \text{Wr}(\mathbf{p}_{\mathbf{t}' \circ \mathbf{s}}; x) \text{Wr}(\mathbf{p}_{\mathbf{t}' \circ \mathbf{u}}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \epsilon_{\mathbf{t}' \circ \mathbf{s}} \wedge \epsilon_{\mathbf{t}' \circ \mathbf{u}} \right)^{\wedge(K-1)} \\ \wedge \int_{\mathbb{R}} \text{Wr}(\mathbf{p}_{\mathbf{t}}; x) d\nu(x) \epsilon_{\mathbf{t}} \wedge \epsilon',$$

Next, we notice that we may extend the sum in the  $K-1$ -wedge product to all basic forms of the form  $\epsilon_{\mathbf{s}} \wedge \epsilon_{\mathbf{u}}$ , since if the range of  $\mathbf{s}$  or  $\mathbf{u}$  has nontrivial intersection with that of  $\mathbf{t}$ , wedging by  $\epsilon_{\mathbf{t}} \wedge \epsilon'$  will cause that term to vanish in the final expression. That is,

$$Z_N \epsilon'_{\text{vol}} = \sum_{\mathbf{t}; \underline{L} \nearrow \underline{NL}} \frac{1}{(K-1)!} \\ \left( \sum_{\mathbf{s}, \mathbf{u}; \underline{L} \nearrow \underline{NL}} \frac{1}{2} \int_{\mathbb{R}^2} \text{Wr}(\mathbf{p}_{\mathbf{s}}; x) \text{Wr}(\mathbf{p}_{\mathbf{u}}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \epsilon_{\mathbf{s}} \wedge \epsilon_{\mathbf{u}} \right)^{\wedge(K-1)} \\ \wedge \int_{\mathbb{R}} \text{Wr}(\mathbf{p}_{\mathbf{t}}; x) d\nu(x) \epsilon_{\mathbf{t}} \wedge \epsilon'.$$

Now, since the  $K-1$ -fold wedge product is independent of  $\mathbf{t}$  we may factor it out of the



sum to write  $Z_N \epsilon'_{\text{vol}}$  as

$$\begin{aligned} & \frac{1}{(K-1)!} \left( \sum_{\mathbf{s}, \mathbf{u}: \underline{L} \nearrow \underline{NL}} \frac{1}{2} \int_{\mathbb{R}^2} \text{Wr}(\mathbf{p}_{\mathbf{s}}; x) \text{Wr}(\mathbf{p}_{\mathbf{u}}; y) \text{sgn}(y-x) d\nu(x) d\nu(y) \epsilon_{\mathbf{s}} \wedge \epsilon_{\mathbf{u}} \right)^{\wedge(K-1)} \\ & \quad \wedge \sum_{\mathbf{t}: \underline{L} \nearrow \underline{NL}} \int_{\mathbb{R}} \text{Wr}(\mathbf{p}_{\mathbf{t}}; x) d\nu(x) \epsilon_{\mathbf{t}} \wedge \epsilon'. \\ & = \frac{1}{(K-1)!} \left( \frac{1}{2} \int_{\mathbb{R}^2} \omega(x) \wedge \omega(y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right)^{\wedge(K-1)} \wedge \int_{\mathbb{R}} \omega(x) \wedge \epsilon' d\nu(x), \end{aligned}$$

where the last equation follows from the definition of  $\omega$ .

Finally, using the binomial theorem,

$$\begin{aligned} & \frac{1}{K!} \left( \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \omega(x) \wedge \omega(y) \text{sgn}(y-x) d\nu(x) d\nu(y) + \int_{\mathbb{R}} \omega(x) \wedge \epsilon' d\nu(x) \right)^{\wedge K} \\ & = \sum_{k=0}^K \frac{1}{(K-k)!k!} \left( \frac{1}{2} \int_{\mathbb{R}^2} \omega(x) \wedge \omega(y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right)^{\wedge(K-k)} \\ & \quad \wedge \left( \int_{\mathbb{R}} \omega(x) \wedge \epsilon' d\nu(x) \right)^{\wedge k}. \end{aligned}$$

However, since every term in this sum must contain exactly one factor of  $\epsilon'$  we see that the terms corresponding to  $k=1$  are the only ones which contribute. That is,

$$\begin{aligned} & \frac{1}{K!} \left( \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \omega(x) \wedge \omega(y) \text{sgn}(y-x) d\nu(x) d\nu(y) + \int_{\mathbb{R}} \omega(x) \wedge \epsilon' d\nu(x) \right)^{\wedge K} \\ & = \frac{1}{(K-1)!} \left( \frac{1}{2} \int_{\mathbb{R}^2} \omega(x) \wedge \omega(y) \text{sgn}(y-x) d\nu(x) d\nu(y) \right)^{\wedge(K-1)} \wedge \int_{\mathbb{R}} \omega(x) \wedge \epsilon' d\nu(x), \end{aligned}$$

which completes the proof of this case.

#### 4.4 Case: $\beta = L^2 + 1$ even, $N$ even

We use the fact that, if  $N = 2M$  and

$$\mathbf{S}(\boldsymbol{\lambda}) = \left[ \frac{(\lambda_n^M - \lambda_m^M)^2}{\lambda_n - \lambda_m} \right]_{m,n=1}^N \quad \text{then} \quad \text{Pf } \mathbf{S}(\boldsymbol{\lambda}) = \prod_{m < n} (\lambda_n - \lambda_m).$$

(See [13, Lemma 5.2]). It follows that

$$\begin{aligned} Z_N &= \frac{1}{N!} \int_{\mathbb{R}^N} \left\{ \prod_{m < n} (\lambda_n - \lambda_m)^{L^2} \right\} \left\{ \prod_{m < n} (\lambda_n - \lambda_m) \right\} d\nu^N(\boldsymbol{\lambda}) \\ &= \frac{1}{N!} \int_{\mathbb{R}^N} \det \mathbf{V}(\boldsymbol{\lambda}) \text{Pf } \mathbf{S}(\boldsymbol{\lambda}) d\nu^N(\boldsymbol{\lambda}). \end{aligned}$$

The proof in this case is then the same as given in Section 4.2 by replacing the particulars of the matrix  $\mathbf{T}(\boldsymbol{\lambda})$  with those of the matrix  $\mathbf{S}(\boldsymbol{\lambda})$ .

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